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INTERPRETATION OF A DYNAMICAL APPROXIMATION FOR ISOTROPIC TURBULENCE*

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1. INTRODUCTION AND SUMMARY

Recently the author has described parts of a theory of homogeneous turbulence which is based upon a new kind of perturbation method. This method leads to closed statistical equations for the velocity covariance upon making a dynamical approximation which was termed the direct-interaction approximation.[1-3]

In the present paper we shall first give a brief review of this theory in which the basic equations will be interpreted dynamically, but not derived. In contrast to the previous treatments, which were devoted primarily to the stationary, isotropic case, we shall be concerned here with freely decaying turbulence.

Apart from the presentation of the statistical equations of the theory for the decaying isotropic case, the new content of the present paper lies in the introduction of a special kind of dynamical model which is statistically related to the actual turbulence system. This model has the property that the equations of the direct-interaction approximation describe it exactly. The existence of such a model leads to the conclusion that these equations have certain important consistency properties. Also, it provides insight into the dynamical meaning and the limitations of the direct-interaction approximation.

2. THE STATISTICAL EQUATIONS OF MOTION

Let us consider an incompressible fluid in a state of isotropic turbulence within a large domain of side L. If we analyze the velocity field $\tilde{u}_i(\underset{\sim}{x},t)$ within the domain by a Fourier series expansion

$$\tilde{u}_i(\underset{\sim}{x},t) = \sum_{\underset{\sim}{k}} u_i(\underset{\sim}{k},t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}, \tag{2.1}$$

where the summation is over all wave vectors allowed by the boundary conditions, equations of motion for the Fourier coefficients can be deduced from the Navier-Stokes equation. In this way we find[4]

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\underset{\sim}{k},t) = -ik_m P_{ij}(\underset{\sim}{k}) \sum_{\underset{\sim}{k}' + \underset{\sim}{k}'' = \underset{\sim}{k}} u_j(\underset{\sim}{k}',t) u_m(\underset{\sim}{k}'',t), \tag{2.2}$$

where ν is the kinematic viscosity and

$$P_{ij}(\underset{\sim}{k}) = \delta_{ij} - k^{-2}k_i k_j. \tag{2.3}$$

The first term of $P_{ij}(\underset{\sim}{k})$ serves to include the Reynolds stresses, and the second term, the pressure. We shall regard the $u_i(\underset{\sim}{k},t)$ as the fundamental dynamic variables of the flow system.*

In discussing turbulence we do not seek an exact description of the exceedingly complicated velocity field but rather the average values of physically interesting functions of the field. The averages can be defined in several ways. We shall adopt here the most customary procedure, in which the averages, to be denoted by $\langle\ \rangle$, are taken over a suitable representative ensemble of individual flows.[4] The statistical quantity of greatest interest is the second-order covariance tensor $\langle u_i(\underset{\sim}{k},t)u_j^*(\underset{\sim}{k},t')\rangle$. It can be shown as a consequence of isotropy that this tensor must have the form

$$(L/2\pi)^3\langle u_i(\underset{\sim}{k},t)u_j^*(\underset{\sim}{k},t')\rangle = \tfrac{1}{2}P_{ij}(\underset{\sim}{k})U(k;t,t') , \tag{2.4}$$

where the scalar U is real, does not depend on the direction of $\underset{\sim}{k}$, and is a symmetric function of t and t'.[4,5] The normalization by $(L/2\pi)^3$ is done so that, in the limit $L \to \infty$, when the spacing of the allowed $\underset{\sim}{k}$ vectors becomes infinitely close, the mean energy per unit mass is given by

$$\tfrac{1}{2}\sum_{\underset{\sim}{k}}\langle u_i(\underset{\sim}{k},t)u_i^*(\underset{\sim}{k},t)\rangle \to \int_0^\infty E(k,t)dk, \tag{2.5}$$

where

$$E(k,t) = 2\pi k^2 U(k;t,t) \tag{2.6}$$

is the energy spectrum function, as usually defined.**

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*In the following, we frequently shall write $u_i(\underset{\sim}{k})$ instead of $u_i(\underset{\sim}{k},t)$ when it is not desired to specify a particular value of the time argument. The same procedure will be followed with other time-dependent quantities also.

**The particular normalization chosen is appropriate to cyclic boundary conditions on the domain. Other usual boundary conditions require minor changes in the definitions.

We shall also define the function

$$r(k;t,t') = \frac{2\pi k^2 U(k;t,t')}{[E(k,t)E(k,t')]^{1/2}}, \tag{2.7}$$

which satisfies $r(k;t,t) = 1$ and measures the phase correlation between the amplitudes of a Fourier mode at times t and t'.

From (2.2) we may obtain the statistical equation of motion

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) U(k;t,t') = S(k;t,t'), \tag{2.8}$$

where

$$S(k;t,t') = (L/2\pi)^3 k_m \sum_{\underset{\sim}{k}' + \underset{\sim}{k}'' = \underset{\sim}{k}} \langle u_i(\underset{\sim}{k}',t) u_m(\underset{\sim}{k}'',t) u_i^*(\underset{\sim}{k},t') \rangle. \tag{2.9}$$

Because of the symmetry of $U(k;t,t')$ in t and t', the similar equation of motion involving $\partial U(k;t,t')/\partial t'$ is redundant with (2.8). For $t = t'$, we find that (2.8) reduces to

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k,t) = T(k,t), \tag{2.10}$$

where

$$T(k,t) = 4\pi k^2 S(k;t,t) \tag{2.11}$$

is the energy transfer function as usually defined.[4] Equation (2.10) expresses the conservation of energy. It exhibits the opposing contributions of viscous dissipation $2\nu k^2 E(k,t)$ and net energy-input from interaction with other modes.

Equation (2.10), or (2.8), cannot be solved directly for $E(k,t)$, or $U(k;t,t')$, because of the presence of the third-order moments on the right sides. Equations of motion for the third-order moments can be obtained by multiplying (2.2) with suitable bilinear expressions and averaging, but as a consequence of the nonlinearity these equations contain fourth-order moments. One does not obtain a closed set of equations for moments of any given orders simply by multiplying (2.2) with various functions and averaging.

Several theories of turbulence have been based on making (2.10) determinate by assuming an expression for the triple moment $T(k,t)$ in terms of $E(k,t)$, obtained usually from some simple analogy for the energy-transfer process together with dimensional considerations. The best-studied example is the eddy-viscosity theory of Heisenberg.[6] Other theories have been based on the assumption that fourth-order moments have the same expression in terms of second-order moments as they would have if the velocity field were normally distributed. In this way a closed set of equations can be obtained which involve only the second- and third-order moments. Examples of this type include the second theory of Heisenberg[6] and the theory of Proudman and Reid.[7]

The approach to be outlined in the present paper is not based on direct surmises about the relations of various moments. Instead, it involves a well-defined approximation on the dynamical processes by which statistical interdependence of the Fourier modes is produced. The closed equations thus obtained turn out to represent the exact behavior of a model system whose dynamical structure is related to that of the actual system. Thereby it can be inferred that the theory obeys important consistency conditions. In formulating this approach it is necessary to consider not only the moments describing the actual statistical state of the system, such as $U(k;t,t')$ and $S(k;t,t')$, but also functions which give the average response of the system to small perturbations. The latter functions embody essential additional aspects of the dynamical behavior.

Let us suppose that at time t' the amplitude of mode $\underset{\sim}{k}$, but of no other mode, is suddenly increased an arbitrary infinitesimal amount $\delta u_i(\underset{\sim}{k},t')$ by some impulsive force. The subsequent history of this perturbation in amplitude - the infinitesimal impulse response - will be very complicated, in general, because of the interaction of mode $\underset{\sim}{k}$ with all the other modes. However, if the response is averaged over the isotropic statistical distribution of the unperturbed amplitudes, we may expect it to become a much simpler function. Therefore we introduce the average impulse-response function $g(k;t,t')$ defined for $t \geq t'$ by

$$\langle \delta u_i(\underset{\sim}{k},t) \rangle = \delta u_i(\underset{\sim}{k},t')g(k;t,t'). \tag{2.12}$$

Note that the averaging implied is only over the unperturbed state, not over the prescribed initial perturbation $\delta u_i(\underset{\sim}{k},t')$. The response indicated by (2.12) is proportional to the initial amplitude jump, despite the nonlinearity of the system, because we are considering only infinitesimal perturbations. In the following, we shall usually call $g(k;t,t')$ simply the response function.

By using (2.2), an equation of motion for $g(k;t,t')$ analogous to (2.8) may be obtained. As in (2.8), the right side of this equation contains higher-order averages than the left, thereby giving rise to similar difficulties. In the present case, the higher averages involve the cross-response of other modes $\underset{\sim}{k}'$ to the initial applied jump in the amplitude of mode $\underset{\sim}{k}$; they are connected with the contribution to the decay of this jump due to spreading of its energy to the other modes. An additional contribution arises, of course, from the viscous dissipation in mode $\underset{\sim}{k}$.

3. THE STRUCTURE OF THE INTERACTION

The nonlinear interaction described by (2.2) is quite complex; each mode interacts with every other mode. However, the total interaction may be considered the resultant of very many elementary interactions of simple structure, each of which involves just three Fourier modes. When $\underset{\sim}{k} = \underset{\sim}{p} + \underset{\sim}{q}$, we shall define the elementary interaction of three modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ by the terms

$$\begin{aligned} &-ik_m P_{ij}(\underset{\sim}{k})\left[u_j(\underset{\sim}{p})u_m(\underset{\sim}{q}) + u_j(\underset{\sim}{q})u_m(\underset{\sim}{p})\right] , \\ &-ip_m P_{ij}(\underset{\sim}{p})\left[u_j(-\underset{\sim}{q})u_m(\underset{\sim}{k}) + u_j(\underset{\sim}{k})u_m(-\underset{\sim}{q})\right] , \\ &-iq_m P_{ij}(\underset{\sim}{q})\left[u_j(\underset{\sim}{k})u_m(-\underset{\sim}{p}) + u_j(-\underset{\sim}{p})u_m(\underset{\sim}{k})\right] \end{aligned} \qquad (3.1)$$

in the equations of motion of the form (2.2) for $u_i(\underset{\sim}{k})$, $u_i(\underset{\sim}{p})$, and $u_i(\underset{\sim}{q})$ respectively. We may represent this elementary interaction by the diagram of Figure 1. It follows from the reality of the velocity field that $u_i(-\underset{\sim}{k}) = u_i^*(\underset{\sim}{k})$. Therefore we shall consider the mode $\underset{\sim}{k}$ to be represented by the amplitude $u_i(-\underset{\sim}{k})$ as well as $u_i(\underset{\sim}{k})$, and we shall include in the definition of elementary interaction the conjugate terms to (3.1) which

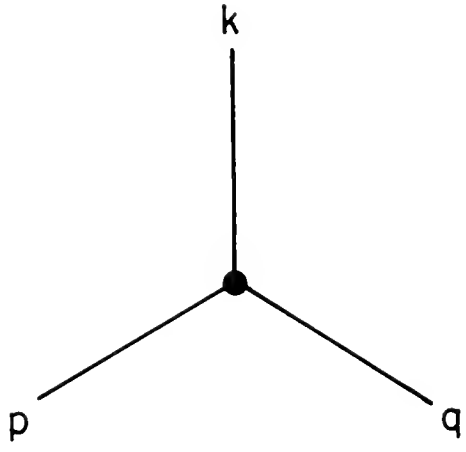

Figure 1. Diagram of the elementary interaction among modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$.

appear in the equations of motion for $u_i(-\underset{\sim}{k})$, $u_i(-\underset{\sim}{p})$, and $u_i(-\underset{\sim}{q})$. It is clear in general that elementary interactions exist only for mode triads whose wave vectors can form a triangle.

The Reynolds stresses and pressure are conservative, so that the nonlinear interaction serves only to distribute the energy among the modes without overall gain or loss. The aptness of the concept of elementary interactions is enhanced by the fact, verifiable from (2.2), that each elementary interaction is individually conservative: the sum of the energy transfers to modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ arising from the terms in (3.1) is zero. Thus the whole process of energy transport may be considered the sum of elemental transfers associated with the individual interactions. The total process is very complicated. Transfer from modes $\underset{\sim}{k}$ and $\underset{\sim}{p}$ to mode $\underset{\sim}{q}$ may take place not only through the elementary interaction which directly links these modes (Figure 1) but also through networks of elementary interactions, involving intermediate modes, which can branch out to extreme complexity (Figure 2). The branching is a graphical expression of the nonlinearity of the equations of motion.

Fortunately, the complexity associated with the multiplicity of transfer paths is compensated by a related feature: each elementary interaction actually represents a very weak dynamical coupling among the modes involved - provided the domain containing the turbulence is very large. Let us consider what happens as the limit $L \to \infty$ is approached. The number-density in wave vector space of the modes allowed by the boundary conditions increases as L^3. Consequently, the number of terms contributing to the right side of (2.2) in any wave vector range increases as L^3. In the limit, each individual term, representing a single elementary interaction, makes only an infinitesimal contribution to the motion of the mode $\underset{\sim}{k}$. This implies that in the limit the effective dynamical coupling among any three individual modes, $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ due to the elementary interaction which directly links them (Figure 1) becomes infinitesimal in strength. The same conclusion is suggested if one takes the terms in the first line of (3.1), which contribute to the motion of $u_i(\underset{\sim}{k})$, and regard them as giving a coupling between the pair of modes $\underset{\sim}{k}$ and $\underset{\sim}{p}$ with the amplitude $\underset{\sim}{u}(\underset{\sim}{q})$ acting as a modulating factor. As $L \to \infty$, the energy-per-unit-mass is spread over an infinitely increasing number of individual modes. Thus the rms value of the amplitude $\underset{\sim}{u}(\underset{\sim}{q})$ becomes infinitesimal, and so does the strength of the pair coupling.

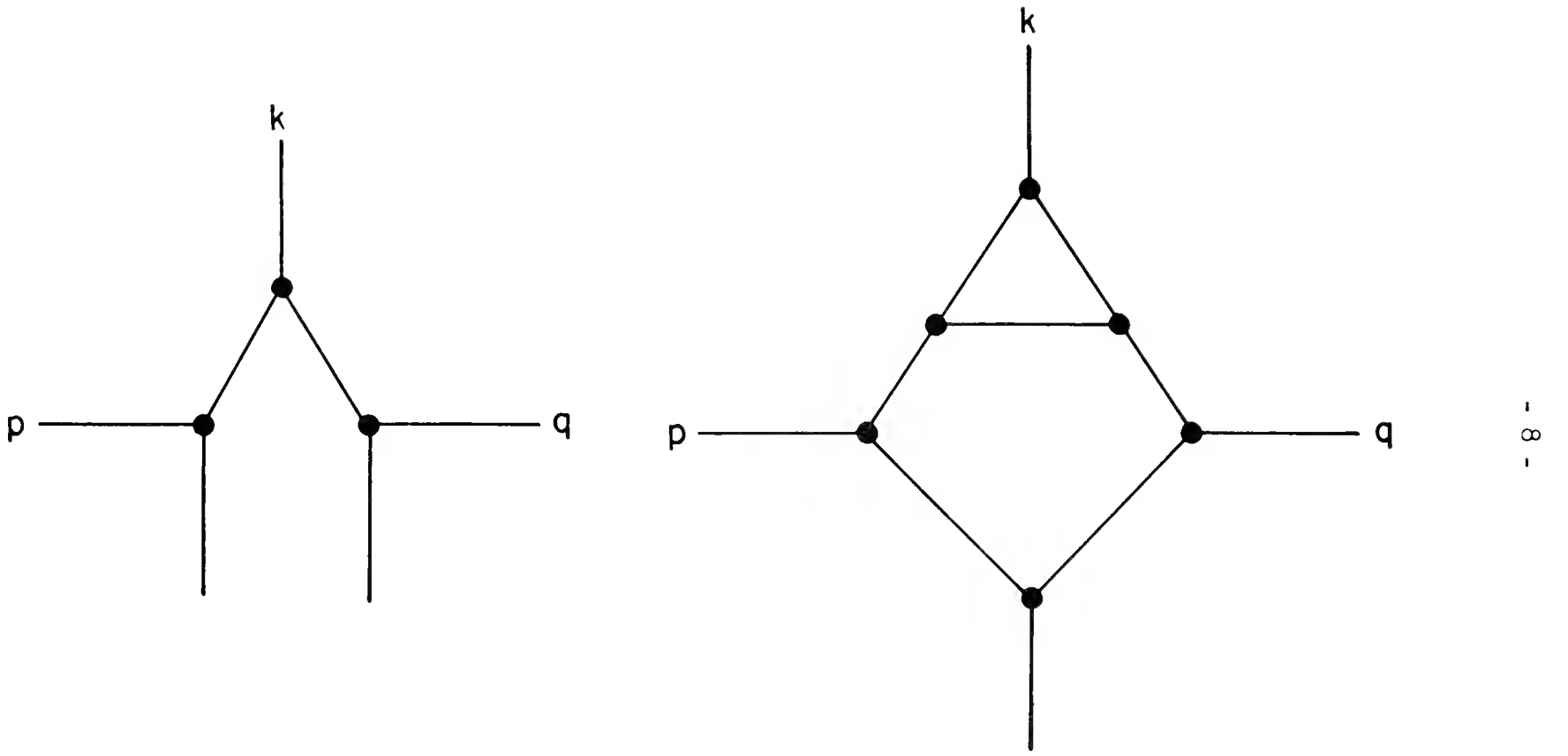

Figure 2. Examples of complex interaction networks linking modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$. The unlabelled lines denote modes other than $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$.

It is not difficult to verify, by keeping account of the orders-of magnitude of relevant quantities as $L \to \infty$, that the weakness of dynamical coupling among any triad of modes is still valid when all the possible paths of coupling involving many elementary interactions and intermediate modes (Figure 2) are taken into account. In fact, the weakness of the total coupling of any three modes is actually a dynamical requirement for the consistency of our underlying assumption of statistical homogeneity. As $L \to \infty$, homogeneity implies that the phase relations among individual Fourier amplitudes become increasingly random,* and this would not be possible if strong dynamic couplings persisted among individual modes. It is extremely important to keep in mind here that these weak couplings do not become negligible as the limit is approached. As they become weaker, there become more of them because there are more allowed modes. Similarly, as the statistical dependencies which they induce among individual modes become weaker, there are more individual cross-moments to be added up in the Fourier sums which represent measurable averages. Indeed, the essential problem of turbulence theory may be considered the evaluation of these cross-moments.

4. THE DIRECT-INTERACTION APPROXIMATION

Let us consider the case where at an initial time t_0 the Fourier amplitudes are all statistically independent and the state of the system is specified by the initial spectrum $E(k,t_0)$. This represents a very considerable idealization of any actual flow, as does our previous assumption of isotropy. However, this initial condition is an especially simple and useful one for studying how the nonlinear interaction induces statistical interdependence of the modes at later times.

We are particularly interested in finding the triple moment

$$\langle u_i(\underset{\sim}{p},t)u_m(\underset{\sim}{q},t)u_i^*(\underset{\sim}{k},t')\rangle \qquad (\underset{\sim}{k} = \underset{\sim}{p} + \underset{\sim}{q}), \tag{4.1}$$

which contributes to $S(k;t,t')$. The most obvious way in which the nonlinear

*See reference 4, Section 2.5.

interaction can induce a nonvanishing value for this moment is through the elementary interaction which directly links the three modes involved (Figure 1). We would also expect the value of the moment to be affected by the more complicated paths of dynamical coupling involving networks of other elementary interactions, and intermediate modes. The distinction just made suggests breaking up the moment into the sum of direct and indirect contributions defined as follows:

Let us remove from the equations of motion of the system the terms (3.1) representing the single elementary interaction directly linking modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$; but let us leave unaltered all the other nonlinear terms, which express the elementary interactions among the rest of the modes and those between $\underset{\sim}{k}$, $\underset{\sim}{p}$, $\underset{\sim}{q}$ and the rest of the modes. We shall call the value of the moment (4.1) induced by these altered equations of motion the indirect contribution. The direct contribution will be defined as the difference between the true value of the moment (with all elementary interaction retained) and the indirect contribution. Thus the direct contribution represents the part of the moment which is induced by the direct elementary interaction acting against the background of all the other elementary interactions.

The approximation fundamental to the present theory is to neglect the indirect contribution to the triple moment. We shall call this the direct-interaction approximation. The dynamical picture underlying it is the following. The elementary interaction directly linking modes $\underset{\sim}{k}$, $\underset{\sim}{p}$ and $\underset{\sim}{q}$ induces an increment in the amplitude of each of these three modes which bears a phase relation to the product of the amplitudes of the other two modes. Thus it yields a contribution to the triple moment. However, this interaction does not take place in isolation. Each of the three modes is coupled to the rest of the system. As a result of this coupling, the induced increments do not simply continue to build up. A relaxation process takes place whereby the energy of the increments is distributed, or mixed, by the overall interaction into many other modes. At the same time, an additional relaxation is caused by the action of viscosity. Thus we have the picture of phase relations among $\underset{\sim}{u}(\underset{\sim}{k})$, $\underset{\sim}{u}(\underset{\sim}{p})$, $\underset{\sim}{u}(\underset{\sim}{q})$ being continually induced by the direct coupling of the three modes and continually broken down by relaxation effects associated with the dynamical interaction as a whole.

Now neglecting the indirect contribution to (4.1) amounts to assuming that the effect of the overall interaction - without the direct interaction - consists entirely of the relaxation process just described. Thus we are ignoring any induction of phase correlation by networks of elementary interactions which can couple modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ only through the agency of intermediate modes. Examples of such networks are shown in Figure 3. It will be noticed that in the diagrams shown each intermediate mode (represented by a line without a free end) is involved in two elementary interactions. This has the effect of eliminating from the associated contribution to (4.1) any dependence on the (random) phases of the amplitudes of the intermediate modes.*

The only justification for the direct-interaction approximation which we can cite at the present point is the wholly intuitive argument that in view of the complexity of the dynamical system the round-about paths of interaction among three modes should be much less effective in inducing definite mutual phase relations than the direct coupling. Later, we shall see that the approximation satisfies important self-consistency properties which strongly indicate its dynamic naturalness and that it actually represents the exact dynamics of a certain model system.

Approximations similar to that described for (4.1) can be defined for moments of fourth and higher orders. We shall not deal with them in the present paper.

It is not very difficult to obtain an exact analytical expression for the direct contribution to (4.1). According to definition, this contribution may be constructed by introducing the elementary interaction of Figure 1 as a perturbation on the equations of motion. Although this single interaction is expected to induce the principal contribution to the triple moment, it is clear from the discussion in the previous Section that actually it can represent only an infinitesimal perturbation of the motion of each of the three modes involved, in the limit $L \to \infty$.** Thus we can express

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*The diagrams in Figure 2 depict examples of more general networks in which some intermediate modes enter only once. Because of the random phases of the intermediate modes, it can be shown, without approximation, that such networks give a vanishing contribution to (4.1) when they are summed over all possible choices of intermediate modes in the limit $L \to \infty$.

**These two properties are consistent since the moment (4.1) represents in toto only an infinitesimal phase correlation among the three modes when $L \to \infty$.

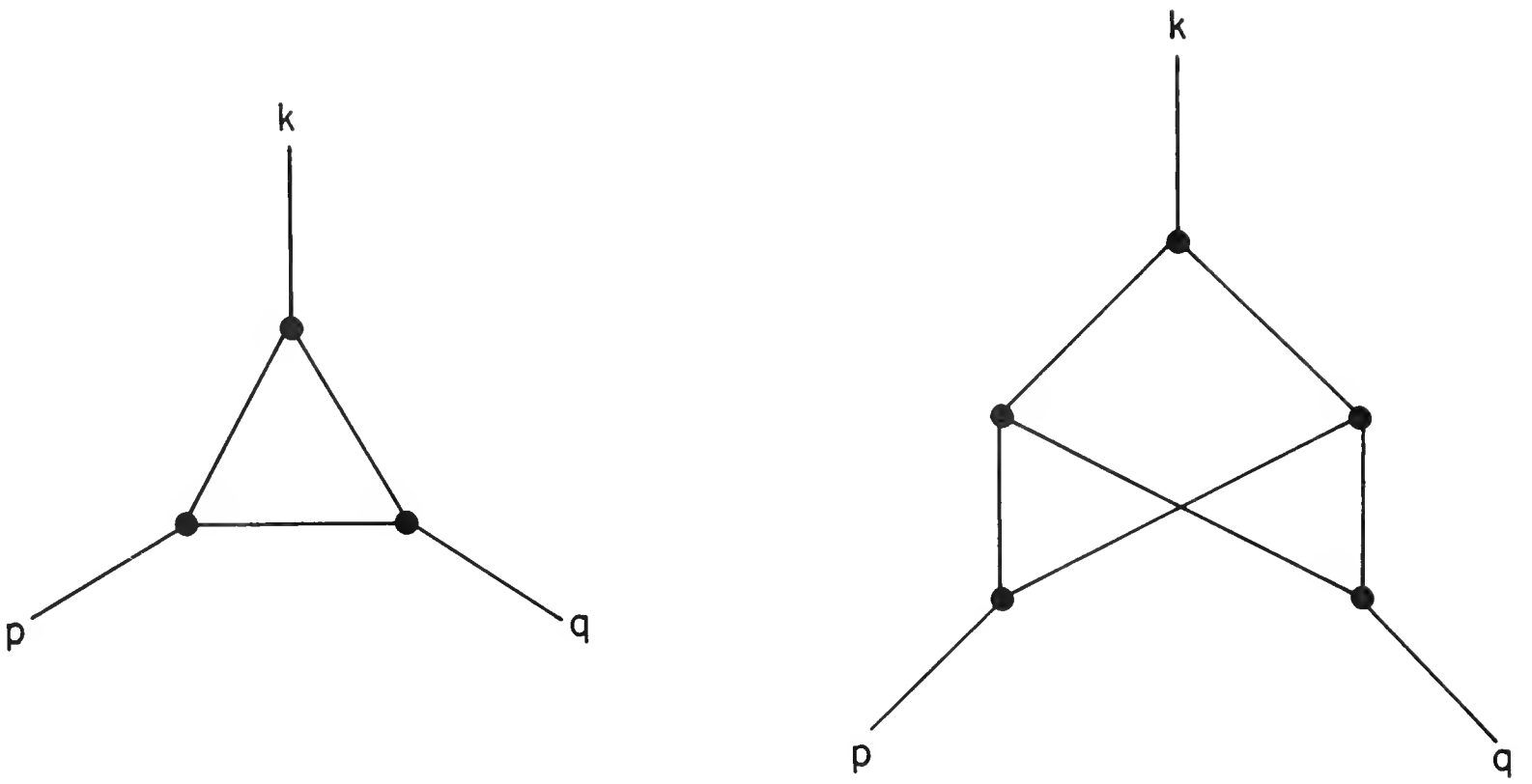

Figure 3. Examples of contributing classes of interaction networks which are neglected in the direct-interaction approximation for the triple moment (4.1) of modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$.

its effects in terms of the response of the modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ to arbitrary infinitesimal disturbances. Then we may average the result over the statistical distribution, taking account of the weakness of the total dynamic coupling and consequent statistical interdependence among the three modes in the limit. After summing the results over $\underset{\sim}{p}$ and $\underset{\sim}{q}$, we arrive at the following exact expression[1,3] for the direct contribution to S(k;t,t'):

$$S(k;t,t') = \pi k \iint_{\triangle} pqdpdq \left\{ \int_{t_0}^{t'} a(k,p,q)g(k;t',s)U(p;t,s)U(q;t,s)ds \right.$$

$$\left. - \int_{t_0}^{t} b(k,p,q)g(p;t,s)U(k;t',s)U(q;t,s)ds \right\}. \qquad (4.2)$$

In this equation the summation over $\underset{\sim}{p}$ and $\underset{\sim}{q}$ has been replaced in the limit by an equivalent integration. The symbols k, p, and q denote wave <u>numbers</u>, and the integration is over the entire domain (denoted by $\triangle$) such that k, p, and q can form the legs of a triangle. The quantities a(k,p,q) and b(k,p,q) are geometrical factors which depend on the shape, but not the size, of this triangle. They are given by

$$a(k,p,q) = \frac{1}{2}(1 - xyz - 2y^2z^2)$$

$$b(k,p,q) = (p/k)(xy + z^3) \qquad (4.3)$$

where x, y, and z are the cosines of the interior angles opposite the legs k, p, q respectively. They obey the identities

$$a(k,p,q) \geq 0$$

$$a(k,p,q) = a(k,q,p)$$

$$k^2 b(k,p,q) = p^2 b(p,k,q)$$

$$b(k,p,q) + b(k,q,p) = 2a(k,p,q) \qquad (4.4)$$

which, as we shall indicate a little later, express important dynamical properties.

In order to facilitate the physical interpretation of (4.2), let us specialize to t' = t. Then we find for the transfer function defined by (2.11),

$$T(k,t) = 4\pi^2 k^3 \iint_{\Delta} pqdpdq \int_{t_0}^{t} [a(k,p,q)g(k;t,s)U(p;t,s)U(q;t,s)$$

$$-b(k,p,q)g(p;t,s)U(k;t,s)U(q;t,s)]ds. \qquad (4.5)$$

The structure exhibited by (4.5) may be rather directly interpreted. The g functions on the right express the decay of infinitesimal perturbations (here associated with the direct interaction) under the influence of viscosity and the overall nonlinear coupling. Let us write each U factor in (4.5) in the form

$$U(k;t,s) = (2\pi k^2)^{-1} r(k;t,s)[E(k,t)E(k,s)]^{1/2}. \qquad (4.6)$$

Now the r functions express the loss of phase auto-correlation in the mode amplitudes under the overall interaction. Thus the appearance of these g and r functions embodies the dynamical relaxation effects discussed above.

It will be noted that the right side of (4.5) is the integral of the difference between two terms. The term containing a(k,p,q) involves the response function for the mode $\underset{\sim}{k}$. As this would suggest, it arises from the induction by the direct interaction of an increment in $\underset{\sim}{u}(\underset{\sim}{k})$ having a phase relation with the product of the amplitudes of modes $\underset{\sim}{p}$ and $\underset{\sim}{q}$. Similarly, the term containing b(k,p,q) involves g(p;t,s) and arises from the induction of an increment in $\underset{\sim}{u}(\underset{\sim}{p})$ phased with the product of the amplitudes of modes $\underset{\sim}{k}$ and $\underset{\sim}{q}$. (Note that we may exchange the roles of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ simply by a change of integration variables.)

Since both the g and r functions express the relaxation effects associated with the overall interaction, we may anticipate that they are either non-negative or have only unimportant negative regions. Now from the first of identities (4.4) we see that a(k,p,q) is non-negative, and the

last of these relations suggests (correctly, it turns out) that $b(k,p,q)$ is typically positive. Thus, the term in (4.5) containing $a(k,p,q)$ plays the role of an absorption term, which always represents a positive flow of energy to mode $\underset{\sim}{k}$, while the term containing $b(k,p,q)$ acts as an emission term. Using (4.6) we find that the absorption term contains the factors

$$[E(p,t)E(p,s)]^{1/2}[E(q,t)E(q,s)]^{1/2}$$

but does not contain $E(k)$.* In contrast, the emission term contains the factors

$$[E(k,t)E(k,s)]^{1/2}[E(p,t)E(p,s)]^{1/2}$$

and thus is linearly proportional to $E(k)$. Consequently, the stronger the excitation in mode $\underset{\sim}{k}$, the more the relative flow out of this mode into modes $\underset{\sim}{p}$ and $\underset{\sim}{q}$. Conversely, the stronger the excitation in modes $\underset{\sim}{p}$ and $\underset{\sim}{q}$, the more the relative flow from them to mode $\underset{\sim}{k}$. It appears that the transfer function (4.5) describes a plausible tendency for the kinetic energy in the various wave numbers to seek some equilibrium through the agency of the nonlinear interaction.

By inserting (4.2) in (2.8) we obtain an equation of motion for $U(k;t,t')$ which involves only the functions g and U. In order to have a complete system we need to obtain a similar equation of motion for $g(k;t,t')$ by a consistent extension of the direct-interaction approximation. This can be done[1,3] without great difficulty, and the result is

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) g(k;t,t') = -\pi k \iint_{\Delta} pq\,dp\,dq\,b(k,p,q) \int_{t'}^{t} g(p;t,s)g(k;s,t')U(q;t,s)\,ds \qquad (4.7)$$

It will be noticed that in contrast to (4.2) the right side here is bilinear in the response functions and linear in the covariance scalars. It is possible

*We use $E(k)$ to denote the spectrum function when it is not desired to specify the value of the time argument.

to give a simple dynamical interpretation of (4.7) which parallels closely the actual derivation.

We can break up the relaxation process described by $g(k;t,t')$ into two parts, conceptually. First, an initial perturbation in mode $\underset{\sim}{k}$ will induce increments in the amplitudes of other modes $\underset{\sim}{p}$. For each mode $\underset{\sim}{p}$ the magnitude of this increment will be proportional to the amplitude of a third mode $\underset{\sim}{q}$ which acts as a modulating factor. Second, there will be a reaction on mode $\underset{\sim}{k}$. The increment induced in the amplitude of mode $\underset{\sim}{p}$ will in turn induce a counter-increment in mode $\underset{\sim}{k}$, again proportional to the amplitude of the third mode $\underset{\sim}{q}$. The counter-increment will on the average be out of phase with the original perturbation in mode $\underset{\sim}{k}$ and thus represent a drain of the perturbation energy out of mode $\underset{\sim}{k}$. This process is represented diagramatically in Figure 4. The arrows indicate the 'signal path' from mode $\underset{\sim}{k}$ to mode $\underset{\sim}{p}$ and back to mode $\underset{\sim}{k}$.

Turning to (4.7) and writing out $U(q;t,s)$ as an explicit covariance according to (2.4), we may interpret the factors in the time-integral as follows. If the amplitude of mode $\underset{\sim}{k}$ is perturbed an infinitesimal amount at time t' then $g(k;s,t')$ represents the average fraction of this perturbation which remains at time s. Together with the amplitude factor in $U(q;t,s)$ which has argument s, it represents the perturbing force applied to mode $\underset{\sim}{p}$ at time s through the elementary interaction directly linking modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$. This force integrated with the response function $g(p;t,s)$ gives the increment induced in mode $\underset{\sim}{p}$ at time t. This increment, together with the remaining amplitude factor in $U(q;t,s)$ (argument t), represents the perturbing force reacting at time t to produce a counter-increment in mode $\underset{\sim}{k}$.

The approximation made in obtaining (4.7) is the neglect of classes of longer paths of action and reaction on mode $\underset{\sim}{k}$ which involve successive transfers of excitation along chains of modes instead of individual transfers to single modes $\underset{\sim}{p}$. Typical neglected contributions are shown in Figure 5. Here again, the arrows trace the 'signal path' from mode $\underset{\sim}{k}$ back to mode $\underset{\sim}{k}$. It should be noted that even in (4.7) the interaction of modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ with all the rest of the modes is implicitly taken into account to the extent of the relaxation effects included in the g and U functions on the right. As in (4.2), we treat the effects of the direct interaction not in isolation but against the background of the rest of the interaction.

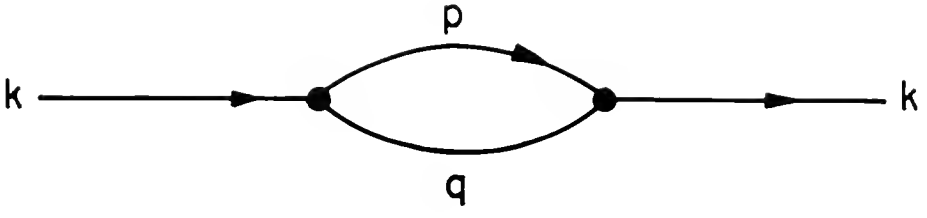

Figure 4. Diagram for the direct-interaction contribution to the response function g(k;t,t').

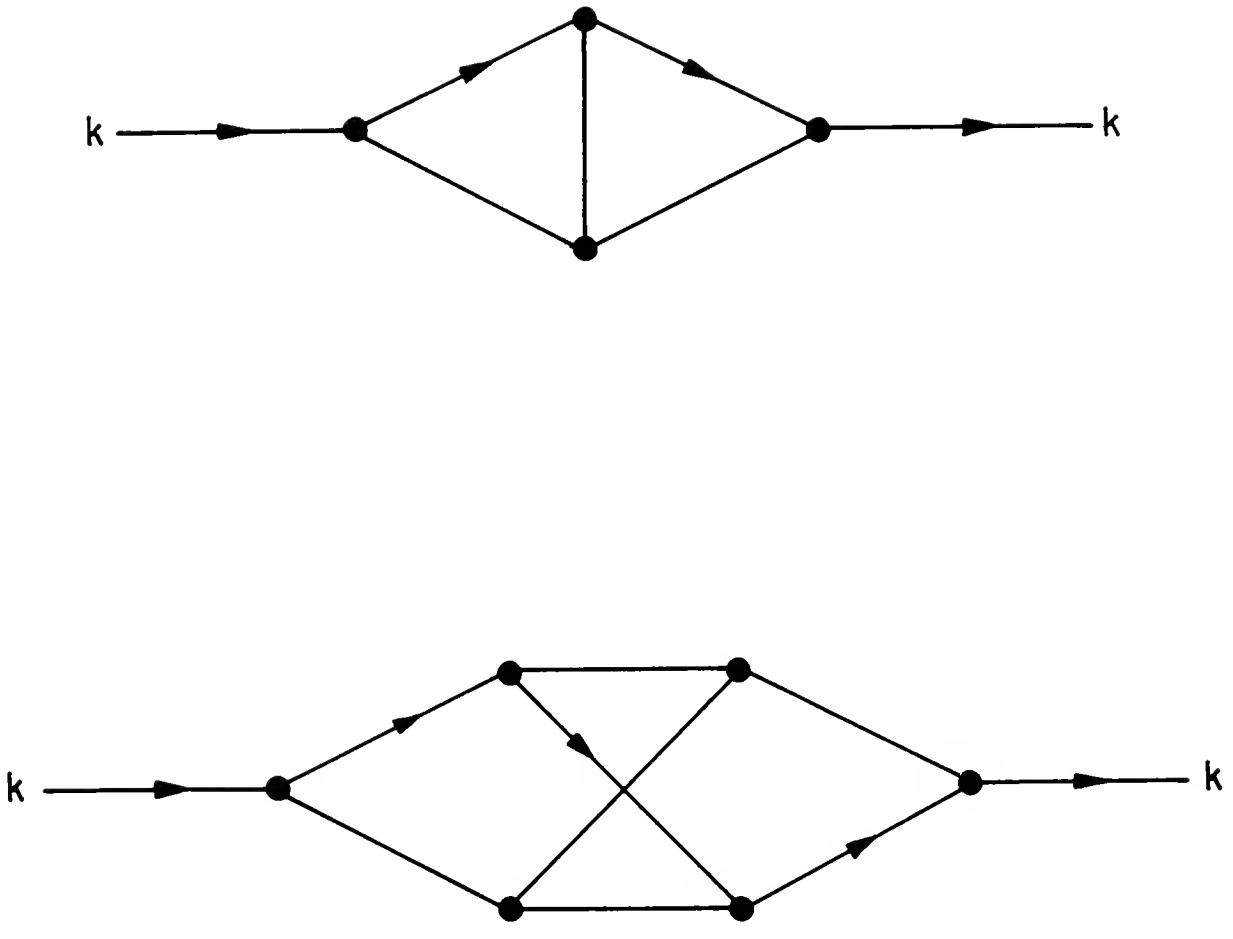

Figure 5. Examples of contributing processes neglected in the direct-interaction approximation for $g(k;t,t')$.

If we append to (2.8), (4.2), and (4.7) the boundary condition

$$g(k;t,t) = 1, \tag{4.8}$$

which according to the definition of $g(k;t,t')$ must hold for all t, we have a complete set of equations which should determine the response and correlation functions once the initial values $E(k,t_0) = 2\pi k^2 U(k;t_0,t_0)$ are prescribed.

5. REPRESENTATION BY A MODEL SYSTEM

Before asking how accurate a picture of turbulence the direct-interaction approximation represents, we have to inquire whether our approximate integro-differential equations lead to any physically meaningful solution at all. This is by no means an academic question. If an arbitrary approximate expression is assumed for $S(k;t,t')$ it is not to be expected that the solution of (2.8) will be the covariance scalar of a possible random process* or that it will go to zero in a physically sensible fashion as $|t - t'| \to \infty$. Furthermore, an arbitrary approximation will not preserve the conservation properties of the interaction discussed in Section3. These observations reflect the general fact that usually we cannot make approximations within a differential or integro-differential equation and obtain a solution which displays the same asymptotic and integral properties as the solution of the exact equation.

It can be seen in a rather simple fashion that the direct-interaction approximation does actually satisfy the consistency requirements we have mentioned. This can be shown by demonstrating that (4.2) and (4.7) are obeyed exactly by a model system in which the coupling of the Fourier modes is altered from that in the real system but the conservation properties of the interaction are preserved. The comparison of this model system with the actual system gives an insight into the real meaning and the domain of validity of our approximation.

Let us consider instead of (2.2) the more general equation

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*For example, $U(k;t,t')$ might turn out to have a frequency spectrum which is negative for some frequencies, thereby implying a negative power spectrum for $u_i(\underset{\sim}{k},t)$.

of motion for $u_i(\underset{\sim}{k},t)$

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\underset{\sim}{k},t) =$$

$$-ik_m P_{ij}(\underset{\sim}{k}) \sum_{\underset{\sim}{k}'+\underset{\sim}{k}''=\underset{\sim}{k}} C(\underset{\sim}{k},\underset{\sim}{k}',\underset{\sim}{k}'')u_j(\underset{\sim}{k}',t)u_m(\underset{\sim}{k}'',t), \qquad (5.1)$$

where the coefficient $C(\underset{\sim}{k},\underset{\sim}{k}',\underset{\sim}{k}'')$ is real, symmetric in $\underset{\sim}{k}$, $\underset{\sim}{k}'$ and $\underset{\sim}{k}''$, and invariant under replacement of any of these three vector arguments by its negative. With this change, (2.8) requires that (2.9) be replaced by the new definition

$$S(k;t,t') =$$

$$(L/2\pi)^3 k_m \sum_{\underset{\sim}{k}'+\underset{\sim}{k}''=\underset{\sim}{k}} C(\underset{\sim}{k},\underset{\sim}{k}',\underset{\sim}{k}'')\langle u_j(\underset{\sim}{k}',t)u_m(\underset{\sim}{k}'',t)u_i^*(\underset{\sim}{k},t')\rangle. \qquad (5.2)$$

It is clear from the symmetry properties of the C's that all the terms in (3.1), representing the elementary interaction linking modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$, are multiplied by the same factor $C(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q})$. From this it follows that the individual conservation property of the elementary interaction is unaltered by the generalization.

Our actual fluid system is represented, of course, by $C(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q}) = 1$ for all $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$. Let us consider instead the new system obtained by letting $C(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q})$ take the value +1 or -1 entirely at random (subject to the symmetry conditions above) when $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ range over the various values allowed by the boundary conditions on our large volume of side L. For this system all the elementary interactions have the same <u>strength</u> as in the original system, but the relative <u>sign</u> of the coefficients of any two (or more) elementary interactions is entirely random.

If one now goes through the direct-interaction approximation for the new system, it is not hard to verify that as a result of the multiplication of (3.1) by $C(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q})$ the expression (in the limit $L \to \infty$) for the

triple moment (4.1) in terms of g and U functions is multiplied by this same factor. Since $[C(\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q})]^2 = 1$, it follows that $S(k;t,t')$, as defined by (5.2), is still given by the expression (4.2). Consequently, the equation of motion for $U(k;t,t')$ is identical for the original and the new systems in the direct-interaction approximation. Similarly, it can be verified that the equation of motion for $g(k;t,t')$ is given by (4.7) for both new and old systems. (This can be inferred from the double appearance of the direct elementary interaction in Figure 4) Thus, the direct-interaction approximation yields identical $g(k;t,t')$ and $U(k;t,t')$ for the new system and the old.

Now let us consider the indirect contributions, neglected by our approximation. Referring to Figures 3 and 5 we note that the indirect contributions necessarily involve several distinct elementary interactions. For the new system the coefficients associated with these interactions are quite randomly related in sign. It is not very hard to see that when one sums over all possible sets of participating interactions the consequence is a random cancellation of the contributions from different sets. In the limit $L \to \infty$, it can be shown that this results in complete suppression of the indirect contributions to $S(k;t,t')$ and the equation of motion for $g(k;t,t')$. This means that for the new system, (2.8), (4.2), and (4.7) actually constitute exact equations of motion for $g(k;t,t')$ and $U(k;t,t')$. We may infer from this the consistency of the direct-interaction approximation in the respects mentioned previously.*

The direct-interaction approximation appears to be the simplest dynamical approximation which embodies the consistency properties in question - that is, realizability of $U(k;t,t')$ as a covariance scalar, proper asymptotic behavior, and detailed energy conservation. Higher approximations with these properties can be constructed also.[3] The next such approximation involves obtaining an approximate expression for fourth-order moments in terms of third-order moments, second-order moments, and higher response functions and using it to close off the set of moment equations. Like the direct-interaction approximation, this leads to integro-differential

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*The conservation properties of the direct-interaction approximation may be verified independently from (4.5) by using the identities (4.4).

equations which are exact for a modified system. Now, however, the modified system bears a closer resemblance to the actual system in that the proper sign relations between different elementary interactions are taken approximately into account. There appear to be a well-defined sequence of successively higher approximations which exactly describe model systems embodying more and more accurately the dynamical structure of the actual system. The higher approximations provide, in principle, a means of estimating the errors associated with the direct-interaction approximation, but they promise severe mathematical difficulties.

The fact that the direct-interaction approximation gives an exact description of our model system suggests the nature of the inaccuracies it generates. As we have noted before, the model system preserves the strengths of all the elementary interactions of mode triads but loses completely the correlation in sign between the coefficients of different elementary interactions. It seems plausible that the mean energy transfer among the modes should depend principally on the relative excitation of the various modes and on the strengths of the interactions which link them. Thus we might expect that it should be fairly well described by the direct-interaction approximation. This surmise is supported by the reasonableness of the dynamical interpretation we have been able to give for the transfer function (4.5). Some further support is given by the application of the theory discussed in the next Section.

Now, however, let us turn to the question of the detailed structure of the turbulence in coordinate space. The evolution of the flow can be partially described as due to the convection, stretching, and twisting of the velocity field by itself. These phenomena seem fairly simple intuitively, but they involve, in essential fashion, the <u>algebraic</u> summation of contributions from all the elementary interactions linking the Fourier components of any given 'eddy structure'. In the model system, consequently, they become scrambled beyond recognition with regard to appearance in coordinate space. This suggests that the direct-interaction approximation should give increasingly poor results when extended to the evaluation of successively higher-order moments, sensitive to the precise spatial form of the velocity structures.

6. APPLICATION TO THE INERTIAL RANGE

Some characteristics and limitations of the direct-interaction approximation may be illustrated by applying it to the inertial range and interpreting the results in terms of the model system introduced in the last section.

Let the rms turbulent velocity in any direction be v_0, and let the rate of dissipation by viscosity per unit mass be ϵ. Then, a wavenumber characterizing the energy-containing range of the turbulence is given[3,4] by

$$k_0 = \epsilon / v_0^3 \tag{6.1a}$$

and a Reynolds number for this range by

$$R_0 = v_0 k_0^{-1} / \nu \tag{6.1b}$$

When $R_0^{1/3} \gg 1$, the integro-differential equations of the direct-interaction approximation simplify greatly for high wavenumbers. Using them, it is found[3] that the inertial range and the range of principal energy dissipation involve wavenumbers k satisfying the inequalities

$$k_0 \ll k \ll R_0 k_0 \ . \tag{6.2}$$

Also, it is found that the transfer of energy is local in wavenumber space, there being no appreciable direct transfer from the energy-containing range to the wavenumbers satisfying (6.2).

The characteristic times for modes satisfying (6.2) are very short compared to the decay time of the turbulence. This has the consequence that $g(k;t,t')$ and $r(k;t,t')$ may be considered explicit functions of only the difference time $t - t'$. The solution of the system (2.8), (4.2), and (4.7) in this range then gives the result[3]

$$g(k;t,t') = r(k;t,t') = \frac{J_1[2v_0 k(t - t')]}{v_0 k(t - t')} \tag{6.3}$$

(It must be remembered here that $g(k;t,t')$ is defined only for $t \geq t'$.) Thus the characteristic time for mode $\underset{\sim}{k}$ is the order of $1/v_0 k$, the time associated with the convection of a structure of this wavenumber by an rms velocity component. The transfer function in the range (6.2) reduces to the form[3]

$$T(k) = k \iint_{\Delta} [k^2 a(k,p,q)E(p) - p^2 b(k,p,q)E(k)]E(q)\,\theta(k,p,q)\frac{dpdq}{pq}, \qquad (6.4)$$

where

$$\theta(k,p,q) = \int_0^{\infty} \frac{J_1(2v_0 ks)}{v_0 ks} \frac{J_1(2v_0 ps)}{v_0 ps} \frac{J_1(2v_0 qs)}{v_0 qs}\, ds$$

$$\sim \frac{(\pi/2)^{1/2}}{v_0(k^2 + p^2 + q^2)^{1/2}} . \qquad (6.5)$$

We have suppressed the time-dependence of T and E in (6.4), since this variation is slow compared to the characteristic times of the modes involved.

The quantity $\theta(k,p,q)$ has the dimensions of a time. In view of our previous discussion of the direct-interaction approximation, we may interpret $\theta(k,p,q)$ as the effective time during which the direct elementary interaction of modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$ can build up phase relations before they are wiped out by the relaxation due to the overall nonlinear interaction, to put it very crudely. In the present case the relaxation evidently is dominated by the action of the energy-containing range, as demonstrated by the factor v_0 in (6.5). Thus, in the direct-interaction approximation the energy-containing region exerts an influence on the rate at which energy is transferred within the high-wavenumber region, even though there is negligible direct transfer from the energy-containing region to the high wavenumbers.

The inertial-range spectrum law in the direct-interaction approximation is easily found from the form of (6.4) and the fact that the energy-transfer turns out to be local in wavenumber space. It is apparent from (6.4) that the rate at which energy is transferred by each elementary interaction involves the spectrum function bilinearly and the velocity v_0 inversely. Since the transfer is local, the rate ϵ at which energy passes from below to above the wavenumber k can depend only on the spectrum function in the neighborhood of k. Thus,

$$\epsilon \sim \frac{k^3}{v_0} [E(k)]^2 , \tag{6.6}$$

where the factor k^3 makes the dimensions of both sides the same, and may be deduced from (6.4).* Inverting (6.6) we have the spectrum law

$$E(k) = \text{const.} \ (\epsilon v_0)^{1/2} k^{-3/2}. \tag{6.7}$$

This law is corroborated, and the constant of proportionality is obtained, by a detailed treatment of (6.4).[3]

It is well-known[4] that according to the Kolmogorov theory the spectrum in the inertail range is given not by (6.7) but by

$$E(k) \propto \epsilon^{2/3} k^{-5/3}. \tag{6.8}$$

The origin of the discrepancy lies in the different roles played by the energy-containing modes in the two theories. Under the direct-interaction approximation, the action of these modes on the high wavenumbers may be described as follows. They induce a rapid (characteristic time $1/v_0 k$) exchange of energy among very many modes, in the neighborhood of a given high $\underset{\sim}{k}$, whose wave vectors differ by the order of k_0. Although these high-lying modes have nearly the same wave vectors, their phases are effectively almost randomly related (cf. Section 3), and so the energy-mixing results in a relaxing of the phase relations, essential for mean energy-transfer, among individual triads consisting of one of the modes in the neighborhood and modes $\underset{\sim}{p}, \underset{\sim}{q}$ in other neighborhoods.

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*If it is assumed that ϵ is proportional to v_0^{-1}, and otherwise is determined by local properties, then (6.6) follows from dimensional considerations alone.

In the Kolmogorov theory, on the other hand, the energy-containing modes have only a trivial convective effect on high k modes and do not directly influence the dynamics in the high k range. On this basis, the energy-containing range should not contribute to the relaxation of energy-transferring phase relations among modes $\underset{\sim}{k}$, $\underset{\sim}{p}$, and $\underset{\sim}{q}$. In fact, if in the expression (6.5) for $\theta(k,p,q)$ we replace v_0 by, say, $[kE(k)]^{1/2}$, which may be considered the rms velocity associated with wavenumbers the order of k only, it may be seen that the modified equation (6.4) thereby obtained leads to the Kolmogorov law (6.8).

It does not seem very easy to decide which of the two inertial range laws is asymptotically correct for infinite R_0, or, for that matter, whether either of them is. The argument basic to the Kolmogorov theory - that the energy-range has only a trivial convective effect on high wavenumbers - is open to doubt. This is because high Reynolds number turbulence tends to display sharp shear fronts[4] which contribute significantly to the high k spectrum, and across which the jump in velocity can be an appreciable fraction of v_0. It is not clear how to separate the low and high wavenumbers in such regions in a physically satisfying fashion. On the other hand, if the Kolmogorov theory is correct, it is not a surprise that the direct-interaction approximation fails to reproduce it. The convection, with only small distortion, of a small-scale velocity structure by a large-scale structure is not a very elementary process in k space. It involves in essential fashion the fact that the coefficients of many elementary interactions linking low wavenumbers with pairs of high wavenumbers are nearly the same. Consequently, as indicated by the discussion in Section 5, it is badly reproduced in the model system which the direct-interaction approximation describes.

The really surprising fact, perhaps, is that the very different dynamical pictures called for by the Kolmogorov theory and the direct-interaction approximation lead to asymptotic laws which are nearly the same. This appears to support the surmise made in the last Section that so far as energy equilibrium is concerned it does not make much difference how the elementary interactions are phased. One might expect, moreover, that the accuracy of the direct-interaction approximation very likely improves at wavenumbers below the inertial range, where convection-without-appreciable-distortion clearly does not occur in any event.

REFERENCES

1) R. H. Kraichnan, Phys. Rev. 109, 1407 (1958); errata: 111, 1747 (1958).

2) R. H. Kraichnan, Phys. Fluids 1, 358 (1958).

3) R. H. Kraichnan, J. Fluid Mech., in press.

4) G. K. Batchelor, The Theory of Homogeneous Turbulence, University Press, Cambridge, 1953.

5) W. C. Meecham, Phys. Fluids 1, 408 (1958).

6) W. Heisenberg, Z. Physik, 124, 628 (1948).

7) I. Proudman and W. H. Reid, Phil. Trans. Roy. Soc. (London) A, 247, 163 (1954).

8) J. Laufer, N.A.C.A. Rep. 1174 (1954).

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